The Bronx High School of Science
Jean Donahue, PhD, Principal
Course: MGN11H – Honors Geometry

Mathematics Department
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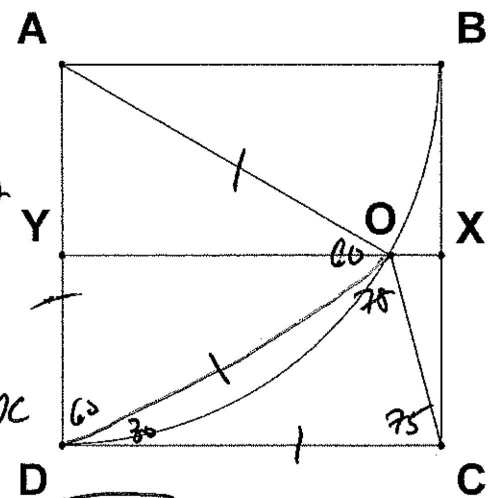
Quadrilateral Problems

1) **(Review/Prelude)** Fill in the chart below with the appropriate answer: "Yes," or "No." Respond "Yes" only if the property is true for *all* quadrilaterals of that type.

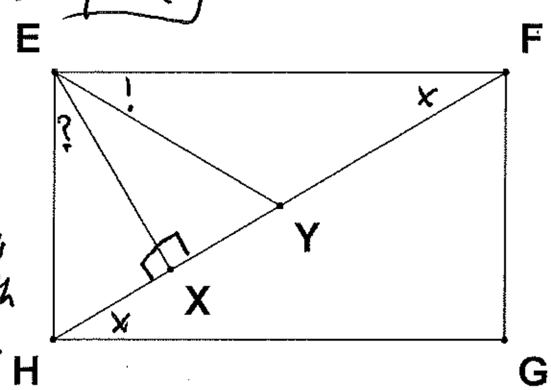
Property	Parallelogram	Rectangle	Rhombus	Square
Opposite sides are congruent.	Yes	Yes	Yes	Yes
Opposite sides are parallel.	Yes	Yes	Yes	Yes
Opposite angles are congruent.	Yes	Yes	Yes	Yes
The diagonals are congruent.	No	Yes	No	Yes
The diagonals bisect each other.	Yes	Yes	Yes	Yes
The diagonals are perpendicular.	No	No	Yes	Yes
All angles are congruent.	No	Yes	No	Yes
All sides are congruent.	No	NO	Yes	Yes

2) **(Normal)** In the diagram at right, $ABCD$ is a square. The circle centered at A with radius AB intersects the perpendicular bisector of $\overline{AD}$ in two points, of which O is the one inside of the square. Compute $m\angle AOC$.



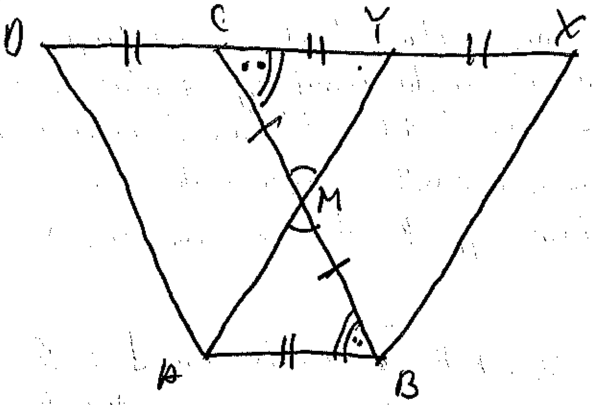
Draw $\overline{OD}$. As O is on $\overline{XY}$, the $\perp$ bisector of $\overline{AD}$, it is equidistant from A and D, so $\overline{AO} \cong \overline{DO}$. As $\overline{AO}$ is a radius of the circle, it is also congruent to $\overline{AB}$ and $\overline{AD}$. Thus, $\overline{AO} \cong \overline{OD} \cong \overline{AD}$, which implies $\triangle AOD$ is equilateral. Thus, $m\angle AOD = m\angle ADO = 60°$. Note $\overline{AB} \cong \overline{DC}$ since they are both sides of the square. Thus, $\triangle ODC$ is isosceles w/ $\overline{OD} \cong \overline{DC}$, and $m\angle ODC = 90-60=30°$. Thus $m\angle DOC = \frac{1}{2}(180-30) = \frac{1}{2}\cdot 150 = 75°$, so $m\angle AOC = 60+75 = \boxed{135°}$.

3) **(Normal)** In the diagram at right, $EFGH$ is a rectangle, with diagonal $\overline{FH}$ drawn. A point X is chosen on $\overline{FH}$ such that $\overline{EX} \perp \overline{FH}$, and Y is the point on $\overline{FH}$ where the diagonals intersect. Compute $m\angle XEH$ and $m\angle YEF$ in terms of x if $m\angle GHF = x$.



The opposite sides $\overline{EF}$ and $\overline{HG}$ are parallel, so $m\angle EFH = x$ as it is an alternate interior angle with $\angle GHF$. $\triangle EFY$ is isosceles because both $\overline{EY}$ and $\overline{FY}$ are half the length of a diagonal, so $\boxed{m\angle YEF = x}$.
Since $\angle EHG$ is right, $m\angle EHX = 90-x$. Thus, in $\triangle EHX$,
$$m\angle XEH = 180-90-(90-x)$$
$$= 90-(90-x) = 90-90+x = x. \quad \boxed{m\angle XEH = x.}$$

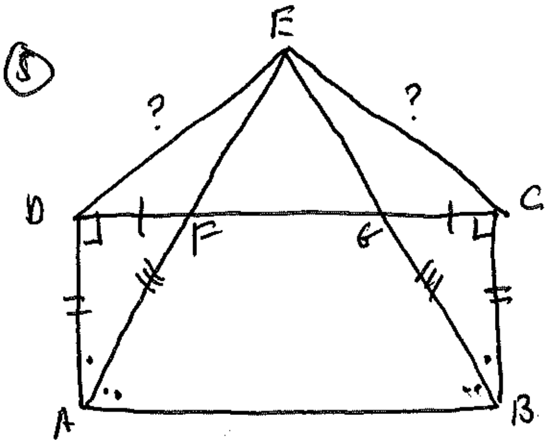
4



Statements	Reasons
1. $\overline{AY}$ bisects $\overline{BC}$ at M.	1. Given.
2. $\angle CMY \cong \angle BMA$ (a≅a)	2. Vertical angles are congruent.
3. M is the midpoint of $\overline{BC}$.	3. Definition of segment bisector. (1)
4. $\overline{CM} \cong \overline{BM}$ (s≅s)	4. Definition of midpoint. (3)
5. $\overline{DCYX}$	5. Given.
6. ABCD is a parallelogram, ABXY is a parallelogram.	6. Given.
* 7. $\overline{AB} \parallel \overline{DC}$, or $\overline{AB} \parallel \overline{DCYX}$	7. Opposite sides of a parallelogram are parallel. (6, 5)
8. $\angle YCM \cong \angle ABM$ (a≅a)	8. If two parallel lines are intersected by a transversal, alternate interior angles are congruent. (7)
9. $\triangle ABM \cong \triangle CYM$	9. ASA Postulate. (2, 4, 8)
10. $\overline{AB} \cong \overline{CY}$	10. Corresponding parts of congruent triangles are congruent. (9)
11. $\overline{AB} \cong \overline{DC}$, $\overline{AB} \cong \overline{YX}$	11. Opposite sides of a parallelogram are congruent. (6)
12. $\overline{DC} \cong \overline{CY} \cong \overline{YX}$	12. Transitive Property. (10, 11)

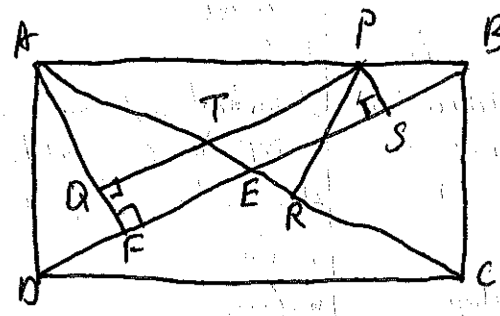
* Note: I used a bit of a shortcut here – while technically $\overline{DCYX}$ isn't a side of ABCD, $\overline{DC}$ is, and as such, the parallelism of opposite sides should extend to the segment as a whole.

5



Statements	Reasons
1. $\overline{DF} \cong \overline{GC}$ (s≅s)	1. Given.
2. $\overline{DFGC}$, $\overline{AFE}$, $\overline{BGE}$	2. Given.
3. ABCD is a rectangle.	3. Given.
4. $\angle ADC$ and $\angle BCD$ are right angles.	4. Each angle in a rectangle is a right angle. (3)
5. $\angle ADC \cong \angle BCD$ (a≅a)	5. All right angles are congruent. (4)
6. $\overline{AD} \cong \overline{BC}$ (s≅s)	6. Opposite sides of a rectangle are congruent. (3)
7. $\triangle ADF \cong \triangle BCG$	7. SAS Postulate. (1, 5, 6)
8. $\angle DAF \cong \angle CBG$ (a≅a)	8. Corresponding parts of congruent triangles are congruent. (7)
9. $\angle DAB$ and $\angle CBA$ are right angles.	9. Same as step 4. (3)
10. $\angle DAB \cong \angle CBA$	10. All right angles are congruent. (9)
11. $\angle DAB - \angle DAF \cong \angle CBA - \angle CBG$ —or— $\angle EAB \cong \angle EBA$	11. Subtraction Postulate. (8, 10)
12. $\overline{AE} \cong \overline{BE}$ (s≅s)	12. If two angles of a △ are ≅, the opposite sides are ≅. (11)
13. $\triangle DAE \cong \triangle CBE$	13. SAS Postulate (6, 8, 12).
14. $\overline{DE} \cong \overline{CE}$	14. Corresponding parts of congruent triangles are congruent. (13)

(6)



The first thing to note is that $PQFS$ is a rectangle. This is true since all but the second perpendicular given tells us that $\angle PSF$, $\angle SFQ$, and $\angle FQP$ are right angles. Thus, the remaining angle $\angle QPS$ is a right angle. Since $PQFS$ is a rectangle, opposite sides are congruent, so $PS = QF$.

The second thing to note is that $\triangle ATQ \cong \triangle PTR$. Since both $\overline{BD}$ and $\overline{PQ}$ are $\perp$ to $\overline{AF}$, they are parallel. Thus, alternate interior angles $\angle APT$ and $\angle ABD$ are congruent. Note that by similar reasoning to #3 on this handout, $\triangle AEB$ is isosceles, so $\angle BAE \cong \angle ABD$. Transitive Property assures us that $\angle APT \cong \angle PAT$. So, $\triangle ATP$ is isosceles, and thus $\overline{AT} \cong \overline{TP}$ since they are sides opposite congruent angles in $\triangle ATP$. Combined with the fact that $\angle AQT$ and $\angle PRT$ are right angles and that $\angle ATQ \cong \angle PTR$ since they are vertical angles, $\triangle ATQ \cong \triangle PTR$ by AAS Theorem. Thus, $AQ = PR$.

So, $PR + PS = AQ + QF = \boxed{AF.}$ QED.