The Bronx High School of Science
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Course: MGS21(H) – (Honors) Geometry

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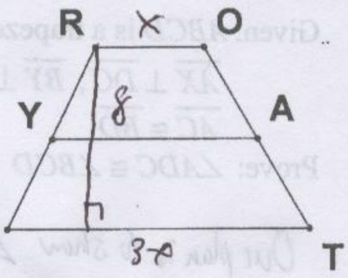
It's Hip to be These—Quadrilaterals Review

1) **(Normal)** In the diagram to the right, *ROTE* is a trapezoid with bases $\overline{RO}$ and $\overline{ET}$, such that $ET = 3 \cdot RO$, and $\overline{RO}$ and $\overline{ET}$ are 8 units apart. If the area of *ROTE* is less than 100, and *RO* and *ET* are integers, compute the maximum length of median $\overline{YA}$.



Let $x = RO$. Thus, $ET = 3x$. We compute the area...

$\frac{1}{2}(8)(x+3x) < 100$.

$4(4x) < 100$

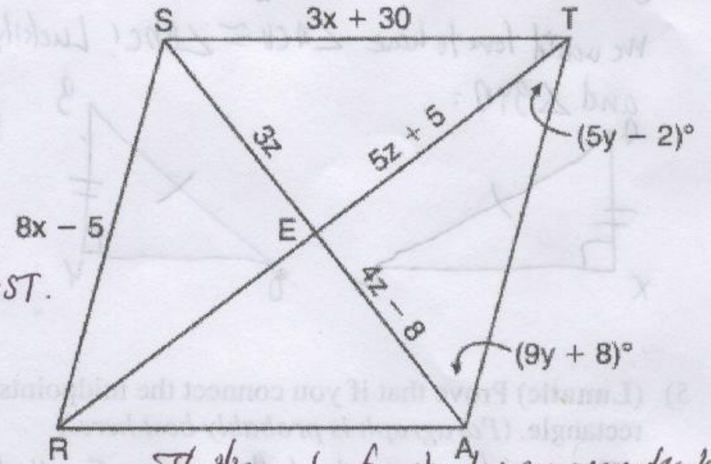
$16x < 100$

$x < \frac{100}{16} = \frac{25}{4}$.

Since x is an integer, the greatest possible value of x is 6. Thus $RO = 6$, $ET = 18$, and median YA is:

$YA = \frac{1}{2}(RO + ET) = \frac{1}{2}(6+18) = \frac{1}{2} \cdot 24 = \boxed{12}$

2) **(Normal)** In the diagram at right, *STAR* is a rhombus with diagonals $\overline{SA}$ and $\overline{TR}$ intersecting at E. $ST = 3x + 30$, $SR = 8x - 5$, $SE = 3z$, $AE = 4z - 8$, $ET = 5z + 5$, $m\angle ETA = 5y - 2$, and $m\angle EAT = 9y + 8$. With this information, compute the numerical value of *SR*, *RT*, and $m\angle TAS$.



Since each side of a rhombus is congruent, we know $RS = ST$.

So: $3x + 30 = 8x - 5$
$-8x - 30 \quad -8x - 30$
$-5x = -35$
$x = 7$.

Thus, $SR = 8 \cdot 7 - 5 = 56 - 5 = \boxed{51}$.

In a rhombus, the diagonals bisect each other. So $SE = EA$, and

$3z = 4z - 8$
$-4z \quad -4z$
$-z = -8$
$z = 8$.

Thus, $ET = 5 \cdot 8 + 5 = 45$, and $RT = 2 \cdot ET = 2 \cdot 45 = \boxed{90}$

The diagonals of a rhombus are perpendicular, so $\angle ETA$ and $\angle EAT$ are complementary. Thus:

$5y - 2 + 9y + 8 = 90$.
$14y + 6 = 90$
$14y = 84 \rightarrow y = 6$. Thus, $m\angle TAS = 9 \cdot 6 + 8 = 54 + 8 = \boxed{62°}$.

3) **(Hard)** The angles of quadrilateral *WXYZ* are such that $m\angle W > m\angle X > m\angle Y > m\angle Z$. These angles are also in *arithmetic progression*—that is, the difference between each angle is constant—so $m\angle W - m\angle X = m\angle X - m\angle Y = m\angle Y - m\angle Z$. If angle *W* measures four times as much as angle *Z*, compute the numerical value of $m\angle W$.

Let $z = m\angle Z$, the smallest angle. Also, we let r be the common difference between angles. Thus, we have

$m\angle Z = z$,
$m\angle Y = z + r$,
$m\angle X = z + 2r$,
$m\angle W = z + 3r$.

By the given information, $m\angle W = 4m\angle Z$, so

$z + 3r = 4z$.
$-z \quad -z$

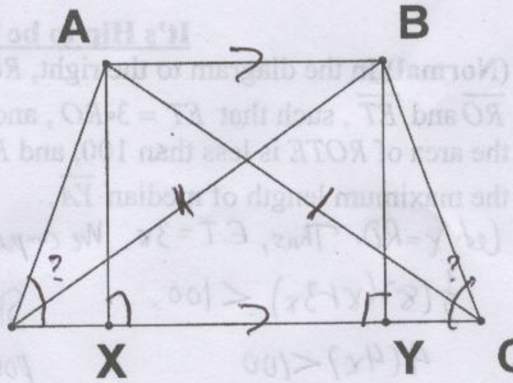
$3r = 3z$
$r = z$.

Huh. The common difference is the same!

So: $m\angle W + m\angle X + m\angle Y + m\angle Z = 360$
$4z + 3z + 2z + z = 360$
$10z = 360$
$z = 36$.

So, $m\angle W = 4 \cdot 36 = \boxed{144°}$.

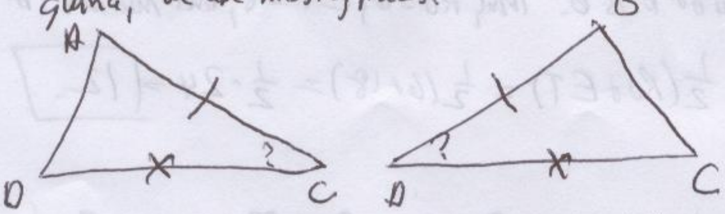
4) **(Hard)** In class, we proved that if a trapezoid is isosceles, then it must have congruent diagonals. However, we took it for granted that the converse is true as well. This, of course, must be rectified. Prove that if a trapezoid has congruent diagonals, then it must be isosceles. Use the following information below:



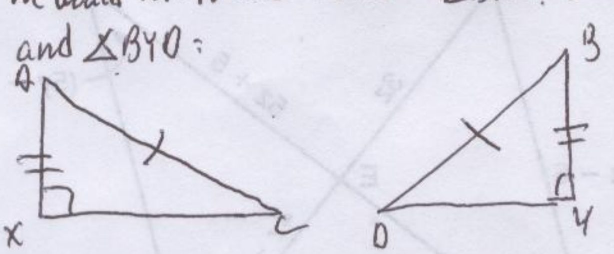
Given: $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$
$\overline{AX} \perp \overline{DC}$, $\overline{BY} \perp \overline{DC}$
$\overline{AC} \cong \overline{BD}$

Prove: $\angle ADC \cong \angle BCD$

Our plan is to show $\triangle ACD \cong \triangle BDC$, but upon first glance, we are missing pieces.

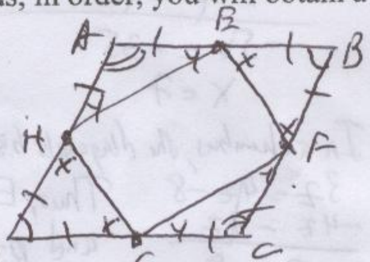


We would love to have $\angle ACD \cong \angle BDC$! Luckily, we can obtain that from another set of triangles; $\triangle AXC$ and $\triangle BYD$:



We use Hyp-leg here, get the pieces we want, and finish the main triangles. Full proof on last page.

5) **(Lunatic)** Prove that if you connect the midpoints of the sides of a rhombus, in order, you will obtain a rectangle. (*Paragraph is probably best here.*)



There are two approaches to this problem - I will show both:

Case I:

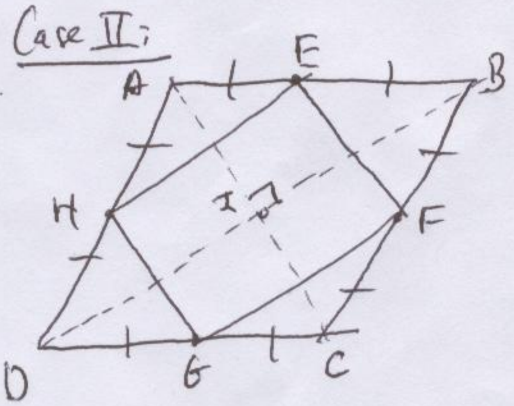
The midpoints divide the segments of the rhombus into two congruent parts. As all of the sides of a rhombus are congruent, each of the individual segments are congruent to each other by Division Postulate. Also, since opposite ~~sides~~ angles of a rhombus are congruent, $m\angle A \cong \angle C$ and $\angle B \cong \angle D$. Thus, we conclude that $\triangle AHE \cong \triangle CGF$ and $\triangle EBF \cong \triangle GDH$ by SAS Postulate. Let $m\angle BEF = x$. Since $\triangle BEF$ is isosceles, $\angle BFE \cong x$. As $\angle DGH$ corresponds to $\angle BEF$ and $\angle DHG$ corresponds to $\angle BFE$, $m\angle DHG = m\angle DGH = x$. Let $m\angle AHE = y$. By the same reasoning, $m\angle AEH = m\angle FGC = m\angle GFC = y$. Thus, if we observe each angle of EFGH, every angle measures $180 - x - y$, as they lie on straight lines with a copy of x and y. Since now, all angles of EFGH are congruent, all angles are right angles, implying EFGH is a rectangle. □

See the last page for an alternate proof.

(4)

Statements	Reasons
1. $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$	1. Given.
2. $\overline{AX} \perp \overline{DC}$, $\overline{BY} \perp \overline{DC}$	2. Given.
3. $\overline{AX} \parallel \overline{BY}$	3. If two lines are perpendicular to the same line, then they are parallel.
4. $ABYX$ is a parallelogram.	4. If a quadrilateral has two pairs of parallel sides, then it is a parallelogram
5. $\angle AXC$ and $\angle BYD$ are right angles.	5. Definition of perpendicular lines.
6. $\triangle AXC$ and $\triangle BYD$ are right triangles.	6. A right triangle contains one right angle.
7. $\overline{AX} \cong \overline{BY}$ (leg $\cong$ leg)	7. Opposite sides of a parallelogram are congruent.
8. $\overline{AC} \cong \overline{BD}$ (hyp $\cong$ hyp)	8. Given.
9. $\triangle AXC \cong \triangle BYD$	9. Hyp-Leg Theorem.
10. $\overline{AC} \cong \overline{BD}$ (s $\cong$ s)	10. Introduced in step 8.
11. $\angle ACD \cong \angle BDC$ (a $\cong$ a)	11. Corresponding parts of congruent triangles are congruent.
12. $\overline{DC} \cong \overline{DC}$ (s $\cong$ s)	12. Reflexive Property.
13. $\triangle ACD \cong \triangle BDC$	13. SAS Postulate.
14. $\angle ADC \cong \angle BCD$	14. Corresponding parts of congruent triangles are congruent.

(5) Case II:



Note in $\triangle ABD$, H and E are midpoints of two sides of the triangle. Thus, $\overline{HE}$ is a midline of $\triangle ABD$, and therefore it is parallel to the third side, $\overline{BD}$. In addition, by the same reasoning in $\triangle BDC$, $\overline{FG} \parallel \overline{BD}$. Thus, by Transitive Property, $\overline{HE} \parallel \overline{FG}$. The same logic applies to $\triangle BCA$ and $\triangle DAC$ to conclude $\overline{EF} \parallel \overline{HG} \parallel \overline{AC}$. Thus, since $EFGH$ has two pairs of parallel sides, $EFGH$ is a parallelogram.

Now, the diagonals of a rhombus are perpendicular. Thus, $\overline{AC} \perp \overline{BD}$. Since $\overline{EH} \parallel \overline{BD}$, we can also conclude that $\overline{EH} \perp \overline{AC}$ since the line is $\perp$ to one of two parallel lines. By the same token, we may conclude that since $\overline{AC} \parallel \overline{EF}$, $\overline{EF} \perp \overline{EH}$ by the same reason as above. Thus, $\angle HEF$ is a right angle. Thus, since $EFGH$ is a parallelogram with a right angle, $EFGH$ is a rectangle. □