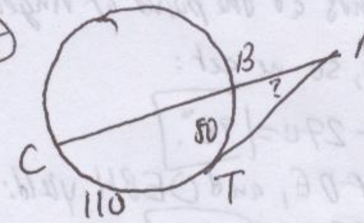
HW #22 Solutions

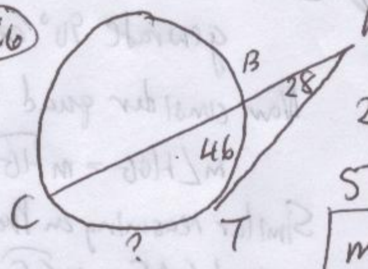
R: p. 360 #15–24, 27
H: p. 360 #15–24, 27, 28, 30

(15)



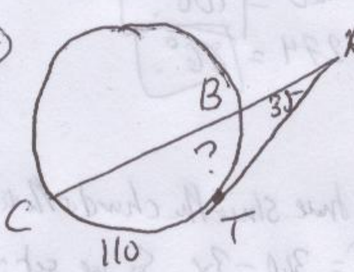
$m\angle A = \frac{1}{2}(m\widehat{CT} - m\widehat{BT})$
$= \frac{1}{2}(110 - 50)$
$= \frac{1}{2}\cdot 60 = \boxed{30^\circ}$

(16)



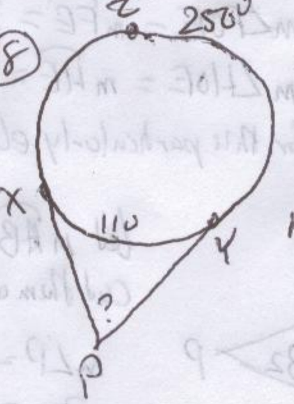
$m\angle A = \frac{1}{2}(m\widehat{CT} - m\widehat{B}$
$28 = \frac{1}{2}(m\widehat{CT} - 46)$
$56 = m\widehat{CT} - 46$
$\boxed{m\widehat{CT} = 102^\circ}$

(17)



$m\angle A = \frac{1}{2}(m\widehat{CT} - m\widehat{BT})$
$35 = \frac{1}{2}(110 - m\widehat{BT})$
$70 = 110 - m\widehat{BT}$
$-40 = -m\widehat{BT}$
$\boxed{40^\circ = m\widehat{BT}}$

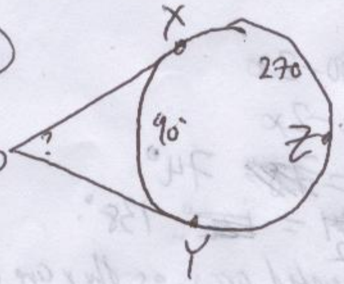
(18)



$m\widehat{XY} = 360 - m\widehat{XZY}$
$= 360 - 250$
$= 110^\circ$

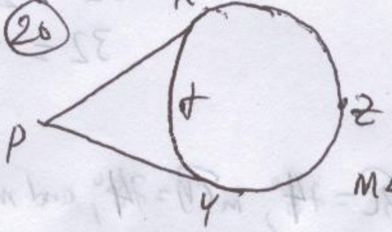
$m\angle P = \frac{1}{2}(m\widehat{XZY} - m\widehat{XY})$
$= \frac{1}{2}(250 - 110)$
$= \frac{1}{2}(140) = \boxed{70^\circ}$

(19)



$m\widehat{XZY} = 360 - 90 = 270^\circ$
$m\angle P = \frac{1}{2}(m\widehat{XZY} - m\widehat{XY})$
$= \frac{1}{2}(270 - 90)$
$= \frac{1}{2}(180) = \boxed{90^\circ}$

(20)



$m\widehat{XZY} = 360 - m\widehat{XY}$
$\boxed{= 360 - t}$

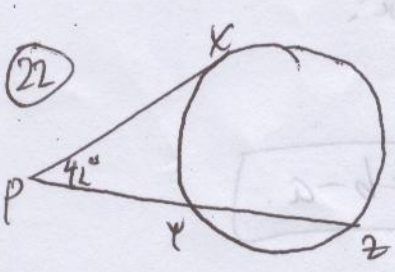
$m\angle P = \frac{1}{2}(m\widehat{XZY} - m\widehat{XY})$
$= \frac{1}{2}(360 - t - t)$
$= \frac{1}{2}(360 - 2t)$
$\boxed{= 180 - t}$

(21)

Let $x = m\widehat{XY}$.
$360 - x = m\widehat{XZY}$.

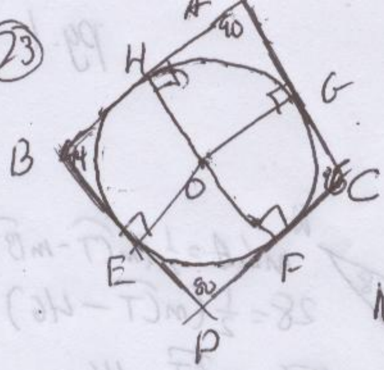
$m\angle P = \frac{1}{2}(m\widehat{XZY} - m\widehat{XY}) \rightarrow \frac{1}{2}(360 - x - x) = 65$
$\frac{1}{2}(360 - 2x) = 65 \longrightarrow 180 - x = 65$
$\boxed{x = 115^\circ}$

(22)



Let ~~x~~ $m\widehat{XY} = 3x$, so $m\widehat{XZ} = 7x$. So:
$m\angle P = \frac{1}{2}(m\widehat{XZ} - m\widehat{XY})$
$42 = \frac{1}{2}(7x - 3x) =$
$42 = \frac{1}{2}\cdot 4x \longrightarrow 42 = 2x \longrightarrow x = 21.$

Thus, ~~$m\widehat{XY} = 21, m\widehat{XZ}$~~
$m\widehat{XY} = 3\cdot 21 = 63$,
$m\widehat{XZ} = 7\cdot 21 = 147$. So,
$m\widehat{YZ} = 360 - (63 + 147)$
$= 360 - 210$
$\boxed{= 150^\circ}$

(23) 

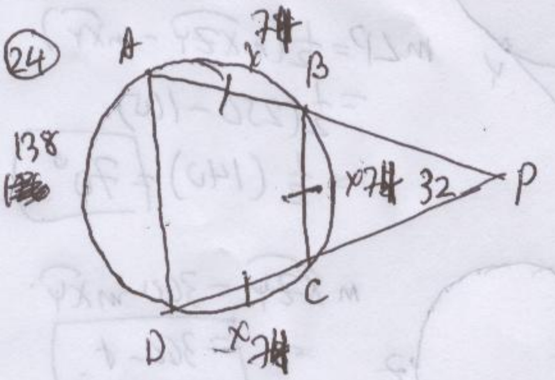
Without loss of generality, let $m\angle A = 90°$, $m\angle B = 94°$, $m\angle C = 96°$, pg. 2
$m\angle D = 80°$. We wish to compute $m\widehat{GH}$, $m\widehat{FG}$, $m\widehat{HE}$, and $m\widehat{EF}$.
Construct the center, O, and draw $\overline{OH}$, $\overline{OG}$, $\overline{OF}$, and $\overline{OE}$. These all generate 90° angles, since radii are $\perp$ to tangents at the point of tangency.
Now, consider quad AHGO. We know 3 of its angles, so we get:

$$m\angle HOG = m\widehat{HG} = 360-(90+90+90) = 360-270 = \boxed{90°}$$

Similar reasoning on the other quadrilaterals OGCF, OFDE, and OEBH yield:

$$m\angle GOF = m\widehat{GF} = 360-(90+90+96) = 360-276 = \boxed{84°}$$
$$m\angle FOE = m\widehat{FE} = 360-(90+90+80) = 360-260 = \boxed{100°}$$
$$m\angle HOE = m\widehat{HE} = 360-(90+90+94) = 360-274 = \boxed{86°}$$

* Credit to Ben Lipkin for this particularly elegant solution!!

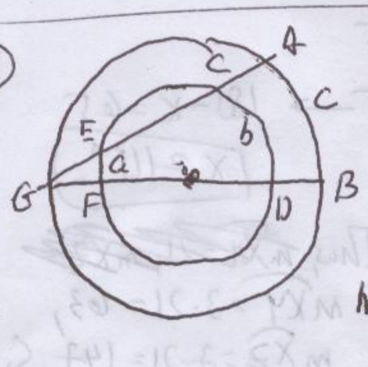
(24) 

Let $m\widehat{AB} = m\widehat{BC} = m\widehat{CD} = x$. This is true since the chords that cut them off are congruent. Thus, $m\widehat{AD} = 360-3x$. So, we get:

$$m\angle P = \tfrac{1}{2}(m\widehat{AD} - m\widehat{BC})$$
$$32 = \tfrac{1}{2}(360-3x-x)$$
$$32 = \tfrac{1}{2}(360-4x) \longrightarrow 32 = 180-2x$$
$$-148 = -2x$$
$$x = 74°$$

Thus, $m\widehat{AB} = 74°$, $m\widehat{BC} = 74°$, $m\widehat{CD} = 74°$, and $m\widehat{AD} = 360-3\cdot 74 = 360-222 = 138°$.
So, the measures of the angles of the quadrilateral are equal to ½ of their intercepted arcs, as they are all inscribed angles. Thus:

$$\boxed{\begin{array}{ll} m\angle A = \tfrac{1}{2}m\widehat{BD} = \tfrac{1}{2}\cdot 2\cdot 74 = 74°. & m\angle C = \tfrac{1}{2}m\widehat{BAD} = \tfrac{1}{2}\cdot(212) = 106°. \\ m\angle B = \tfrac{1}{2}m\widehat{AC} = \tfrac{1}{2}(74+138) = 106°. & m\angle D = \tfrac{1}{2}m\widehat{ABC} = \tfrac{1}{2}\cdot 2\cdot 74 = 74°. \end{array}}$$

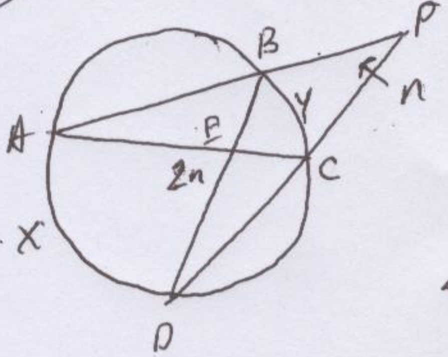
(27) 

Note that $\angle AGB$ is an inscribed angle. Thus, $m\angle AGB = \frac{1}{2}m\widehat{AB}$.
But, $\angle AGB$ is formed by two secants of the inner circle! Thus,

$$m\angle AGB = \tfrac{1}{2}(m\widehat{CD} - m\widehat{EF}).$$ We equate these:

$$m\angle AGB = \tfrac{1}{2}m\widehat{AB} = \tfrac{1}{2}(m\widehat{CD} - m\widehat{EF})$$
$$\tfrac{1}{2}\cdot c = \tfrac{1}{2}(b-a) \longrightarrow \boxed{c = b-a.}$$

(28)



Our goal is to determine $\frac{x}{y}$. Note that $\angle AED$ is an angle formed by two intersecting chords, so:

$$m\angle AED = \frac{1}{2}(m\widehat{AD} + m\widehat{BC})$$

$$2n = \frac{1}{2}(x+y) \longrightarrow x+y=4n.$$

$\angle APD$ is an angle formed by two secants. Thus:

$$m\angle APD = \frac{1}{2}(m\widehat{AD} - m\widehat{BC})$$

$$n = \frac{1}{2}(x-y) \longrightarrow x-y=2n.$$

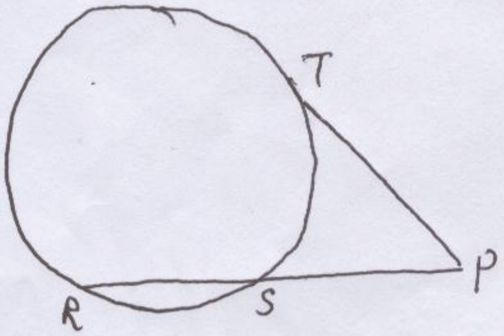
Solve this system simultaneously:

$$\begin{aligned} x+y&=4n \\ +\ x-y&=2n \\ \hline 2x&=6n \rightarrow x=3n. \end{aligned}$$

$$\begin{aligned} x-y&=2n \\ 3n-y&=2n \\ y&=n. \end{aligned}$$

So, $\frac{x}{y} = \frac{3n}{n} = \boxed{\frac{3}{1}}$.

(30)



$\angle P$ is an angle formed outside a circle by a tangent and secant. Thus,

$$m\angle P = \frac{1}{2}(m\widehat{RT} - m\widehat{ST}).$$

~~At minimum~~, we know $80 < m\widehat{ST} < 90$. Also, we can determine that at minimum, $m\widehat{RT}$ and $m\widehat{ST} \simeq 80°$, so the maximum value of $m\widehat{RT} = 360 - 2\cdot 80 = 360 - 160 = 200°$. The minimum value of $m\widehat{RT}$ is attained when $\widehat{RS}$ and $\widehat{ST}$ are largest, so $m\widehat{RS} = m\widehat{ST} = 90°$. Thus, the min value of $m\widehat{RT} =$

$360 - 2\cdot 90 = 180°$. So:

$80 < m\widehat{ST} < 90$ and

$180 < m\widehat{RT} < 200$.

To minimize $\angle P$, we wish for $m\widehat{RT}$ to be small and $m\widehat{ST}$ to be large. Thus, choose $m\widehat{RT} = \cancel{200}\ 180$ and $m\widehat{ST} = 90$. So: $m\angle P = \frac{1}{2}(180-90) = \frac{1}{2}\cdot 90 = 45°$.

To maximize $\angle P$, choose $m\widehat{RT}$ to be large and $m\widehat{ST}$ to be small. So, $m\widehat{RT} = 200$ and $m\widehat{ST} =$

So: $m\angle P = \frac{1}{2}(200-80) = \frac{1}{2}(120) = 60°$.

So, $\boxed{45° < m\angle P < 60°}$.